

GRÖBNER-SHIRSHOV BASES METHOD FOR GELFAND-DORFMAN-NOVIKOV ALGEBRAS*

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Abstract: We establish Gröbner-Shirshov bases theory for Gelfand-Dorfman-Novikov algebras over a field of characteristic 0. As applications, a PBW type theorem in Shirshov form is given and we provide an algorithm for solving the word problem of Gelfand-

*Supported by the NNSF of China (11171118, 11571121) and the Program on International Cooperation and Innovation, Department of Education, Guangdong Province (2012gjh0007).

[†]Supported by Russian Science Foundation (project 14-21-00065).

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Dorfman-Novikov algebras with finite homogeneous relations. We also construct a subalgebra of one generated free Gelfand-Dorfman-Novikov algebra which is not free.

Key words: Gröbner-Shirshov basis; Gelfand-Dorfman-Novikov algebra; commutative differential algebra; word problem.

AMS 2010 Subject Classification: 17D25, 16S15, 13P10, 08A50

1 Introduction

Gelfand-Dorfman-Novikov algebras were introduced by I.M. Gelfand, I.Ya. Dorfman [20], 1979 in connection with Hamiltonian operators in the formal calculus of variations and A.A. Balinskii, S.P. Novikov [4], 1985 in connection with linear Poisson brackets of hydrodynamic type. As it was pointed out in [26], 1985, E.I. Zelmanov answered to a Novikov's question about simple finite dimensional Gelfand-Dorfman-Novikov algebras over a field of characteristic zero at the same year. He proved that there are no such non-trivial algebras, see [45], 1987. In 1989, V.T. Filippov found first examples of simple infinite dimensional Gelfand-Dorfman-Novikov algebras of characteristic $p \geq 0$ and simple finite dimensional Gelfand-Dorfman-Novikov algebras of characteristic $p > 0$, see [18]. J.M. Osborn [28, 29, 30], 1992-1994 gave the name Novikov algebra (he knew both papers [4, 20]) and began to classify simple finite dimensional Gelfand-Dorfman-Novikov algebras with prime characteristic $p > 0$ and infinite dimensional ones with characteristic 0, as well as irreducible modules [31, 32], 1995. Considering the contribution of Gelfand and Dorfman to Novikov algebras, we call Novikov algebras as Gelfand-Dorfman-Novikov algebras in this paper. There are also quite a few papers on the structure theory (see, for example, X. Xu [41, 42, 43, 44], 1995-2000, C. Bai and D. Meng [1, 2, 3], 2001, L. Chen, Y. Niu and D. Meng [14], 2008, D. Burde and K. Dekimpe [11], 2006) and combinatorial theory of Gelfand-Dorfman-Novikov algebras, and irreducible modules over Gelfand-Dorfman-Novikov algebras, with applications to mathematics and mathematical physics. The present paper is on combinatorial method of Gröbner-Shirshov bases for Gelfand-Dorfman-Novikov algebras. Let us observe some combinatorial results. In [20], it was given an important observation by S.I. Gelfand that any differential commutative associative algebra is a Gelfand-Dorfman-Novikov algebra under the new product

$a \circ b = (Da)b$. This observation leads to a notion of the universal enveloping of a Gelfand-Dorfman-Novikov algebra (see and cf. [16]) that we use in the present paper. V.T. Filippov [19], 2001 proved that any Novikov nil-algebra of nil-index n with characteristic 0 is nilpotent (an analogy of Nagata-Higman theorem). He used essentially the Zelmanov theorem [46], 1988 that any Engel Lie algebra of index n in characteristic 0 is nilpotent. By the way, Zelmanov [47], 1989 also proved the local nilpotency of any Engel Lie algebra of index n in any characteristic. In 2002, A. Dzhumadil'daev and C. Löfwall [16] found structure of a free Gelfand-Dorfman-Novikov algebra using trees and a free differential commutative algebra. We use essentially this result here. Dzhumadil'daev [15], 2011 found another linear basis of a free Gelfand-Dorfman-Novikov algebra using Young tableaux. This result was essentially used by L. Makar-Limanov and U. Umirbaev [25] in a proof of the Freiheitssatz theorem for Gelfand-Dorfman-Novikov algebras. Also they proved that the basic rank of the variety of Gelfand-Dorfman-Novikov algebras is one.

In the present paper, we introduce Gröbner-Shirshov bases method for Gelfand-Dorfman-Novikov algebras, prove a PBW type theorem in Shirshov form for Gelfand-Dorfman-Novikov algebras, and provide an algorithm for solving the word problem for Gelfand-Dorfman-Novikov algebras with finite number of homogeneous relations.

Gröbner and Gröbner-Shirshov bases methods were invented by A.I. Shirshov, a student of A.G. Kurosh, for Lie algebras and implicitly for associative algebras [37], 1962 and non-associative (commutative anti-commutative) algebras [36], 1954, by H. Hironaka for commutative topological algebras [21], 1964 and by B. Buchberger [10], 1965 for commutative algebras. As a prehistory, see A.I. Zhukov [48], 1950, another Kurosh's student.

Gröbner-Shirshov (Gröbner) bases methods deal with varieties and categories of (differential, integro-differential, PBW, Leavitt, Temperley-Lieb, Iwahori-Hecke, quadratic, free products of two, over a commutative algebra, ...) associative algebras, (plactic, Chines, inverse, ...) semigroup algebras, (Coxeter, braid, Artin-Tits, Novikov-Boone, ...) group algebras, semiring algebras, Lie (restricted, super-, semisimple, Kac-Moody, quantum, Drinfeld-Kohno, over a commutative algebra, metabelian, ...) algebras, associative conformal algebras, Loday's (Leibniz, di-, dendriform) algebras, Rota-Baxter algebras, pre-Lie (i.e., right symmetric) algebras, (simplicial, strict monoidal, ...) cate-

gories, non-associative (commutative, anti-commutative, Akiwis, Sabinin, ...) algebras, symmetric (non-symmetric) operads, Ω -algebras, modules, and so on. Gröbner-Shirshov bases method is useful in homological algebra (Anick resolutions, (Hochschild) cohomology rings of (Leavitt, plactic, ...) algebras), in proofs of PBW type theorems (Lie algebra – associative algebra, Lie algebra – pre-Lie algebra, Leibniz algebra – associative dialgebra, Akiwis algebra – non-associative algebra, Sabinin algebra - modules), in algorithmic problems of algebras (solvable and unsolvable algorithmic problems), in the theory of automatic groups and semigroups, in independent constructions of Hall, Hall-Shirshov and Lyndon-Shirshov bases of a free Lie algebra, on embedding theorems and many other applications. For details one may see, for example, new surveys [6] and [34].

In this paper we prove a PBW type theorem in Shirshov form for Gelfand-Dorfman-Novikov algebras. The first of this kind of theorems is the following (see [7, 8, 9]).

Let $L = Lie(X|S)$ be a Lie algebra, presented by generators X and defining relations S over a field k , $U(L) = k\langle X|S^{(-)} \rangle$ the universal enveloping associative algebra of L (here $S \Rightarrow S^{(-)}$ using $[x, y] \Rightarrow xy - yx$). Then S is a Lie Gröbner-Shirshov basis in $Lie(X)$ if and only if $S^{(-)}$ is an associative Gröbner-Shirshov basis in $k\langle X \rangle$.

As a corollary, let S be a Lie Gröbner-Shirshov basis (in particular, S be a multiplication table of L). Then

(i) A linear basis of $U(L)$ consists of words $u_1 u_2 \dots u_k$, $k \geq 0$, where u_i 's are $S^{(-)}$ -irreducible associative Lyndon-Shirshov words (without brackets) in X , $u_1 \leq u_2 \leq \dots \leq u_k$ in lexicographical order (meaning $a > ab$, if $b \neq 1$) (in particular, a linear basis of $U(L)$ is PBW one if S is a multiplication table of L).

(ii) A linear basis of $U(L)$ consists of words $[u_1][u_2] \dots [u_k]$, $k \geq 0$, where $[u_i]$'s are S -irreducible Lyndon-Shirshov Lie words in X , $u_1 \leq u_2 \leq \dots \leq u_k$ in lexicographical order.

(iii) A linear basis of L consists of words $[u]$, where $[u]$'s are S -irreducible Lyndon-Shirshov Lie words in X .

For Gelfand-Dorfman-Novikov algebras we prove the following PBW type theorem in Shirshov form.

Let $GDN(X)$ be a free Gelfand-Dorfman-Novikov algebra, $k\{X\}$ be a free commutative differential algebra, $S \subseteq GDN(X)$ and S^c a Gröbner-Shirshov basis in $k\{X\}$, which

is obtained from S by Buchberger-Shirshov algorithm in $k\{X\}$. Then

(i) $S' = \{uD^m s \mid s \in S^c, u \in [D^\omega X], wt(u\overline{D^m s}) = -1, m \in \mathbb{N}\}$ is a Gröbner-Shirshov basis in $GDN(X)$.

(ii) The set $Irr(S') = \{w \in [D^\omega X] \mid w \neq u\overline{D^t s}, u \in [D^\omega X], t \in \mathbb{N}, s \in S^c, wt(w) = -1\} = GDN(X) \cap Irr[S^c]$ is a linear basis of $GDN(X|S)$. Thus, any Gelfand-Dorfman-Novikov algebra $GDN(X|S)$ is embeddable into its universal enveloping commutative differential algebra $k\{X|S\}$.

Using Buchberger-Shirshov algorithm, we provide algorithms for solving both the word problem for commutative differential algebras with finite number of $D \cup X$ -homogeneous defining relations and the word problem for Gelfand-Dorfman-Novikov algebras with finite number of X -homogeneous defining relations. For Lie algebras it was proved by Shirshov in his original paper [37], see also [38]. In general, word problem for Lie algebras is unsolvable, see [5]. For Gelfand-Dorfman-Novikov algebras it remains unknown. So far, the word problem (membership problem) for commutative differential algebras is solved mainly for the following cases [23]: radical ideals, isobaric (i.e., homogeneous with respect to derivations) ideals, ideals with a finite or parametrical standard basis, and ideals generated by a composition of two differential polynomials (under some additional assumptions).

Finally, we prove that the variety of Gelfand-Dorfman-Novikov algebras is not a Schreier one, i.e., not each subalgebra of a free Gelfand-Dorfman-Novikov algebra is free. The most famous Schreier variety are the variety of groups [33], the variety of non-associative algebras [24], the variety of (non-associative) commutative and anti-commutative algebras [36], the variety of Lie algebras [35, 40]. For more details, see [13, 39].

2 Free Gelfand-Dorfman-Novikov algebras

A non-associative algebra $A = (A, \circ)$ is called a right-Gelfand-Dorfman-Novikov algebra [15], if A satisfies the identities

$$x \circ (y \circ z) - (x \circ y) \circ z = x \circ (z \circ y) - (x \circ z) \circ y,$$

$$x \circ (y \circ z) = y \circ (x \circ z).$$

In the papers [15, 16], the authors constructed the free Gelfand-Dorfman-Novikov algebra $GDN(X)$ generated by X as follows: A Young diagram is a set of boxes with non-increasing numbers of boxes in each row. Rows and columns are numbered from top to bottom and from left to right. Let p be the number of rows and r_i be the number of boxes in the i th row. To construct Gelfand-Dorfman-Novikov diagram, we need to complement Young diagram by one box in the first row. To construct Gelfand-Dorfman-Novikov tableaux on a well-ordered set X , we need to fill Gelfand-Dorfman-Novikov diagrams by elements of X . Denote by $a_{i,j}$ an element of X in the box that is the cross of the i th row by the j th column. The filling rule is the following:

- (a) $a_{i,1} \geq a_{i+1,1}$, if $r_i = r_{i+1}$, $i = 1, 2, \dots, p-1$;
- (b) The sequence $a_{p,2} \cdots a_{p,r_p} a_{p-1,2} \cdots a_{p-1,r_{p-1}} \cdots a_{1,2} \cdots a_{1,r_1+1}$ is non-decreasing.

Such a Gelfand-Dorfman-Novikov tableau corresponds to the following element of the free Gelfand-Dorfman-Novikov algebra:

$$\begin{aligned} w &= Y_p \circ (Y_{p-1} \circ (\cdots \circ (Y_2 \circ Y_1) \cdots)) \quad (\text{right-normed bracketing}), \text{ where} \\ Y_i &= (\cdots ((a_{i,1} \circ a_{i,2}) \circ a_{i,3}) \cdots \circ a_{i-1,r_{i-1}}) \circ a_{i,r_i}, \quad 2 \leq i \leq p, \\ Y_1 &= (\cdots ((a_{1,1} \circ a_{1,2}) \circ a_{1,3}) \cdots \circ a_{1,r_1}) \circ a_{1,r_1+1} \end{aligned}$$

(each Y_j left-normed bracketing). In this case, we say w has degree $r_p + r_{p-1} + \cdots + r_1 + 1$. We call such a w as a Gelfand-Dorfman-Novikov tableau as well. Such elements form a linear basis of a free Gelfand-Dorfman-Novikov algebra generated by X and we denote such free Gelfand-Dorfman-Novikov algebra as $GDN(X)$, see [15, 16].

A commutative differential algebra $A = (A, \cdot, D)$ is a commutative associative algebra with one linear operator $D : A \rightarrow A$ such that for any $a, b \in A$, $D(ab) = (Da)b + a(Db)$. We call such a D a derivation of A .

Given a well-ordered set $X = \{a, b, c, \dots\}$, denote $D^\omega X = \{D^i a \mid i \in \mathbb{N}, a \in X\}$, $[D^\omega X]$ the free commutative monoid generated by $D^\omega X$ and k a field of characteristic 0. Let $D(1) = 0, D^0 a = a, D(D^i a) = D^{i+1} a, D(\alpha u + \beta v) = \alpha Du + \beta Dv$ and $D(uv) = (Du) \cdot v + u \cdot D(v)$ for any $a \in X, \alpha, \beta \in k, u, v \in [D^\omega X]$ (\cdot is often omitted). Then $(k[D^\omega X], \cdot, D)$ is a free commutative differential algebra over k , see [22]. From now on

we denote a as $a[-1]$, $D^{i+1}a$ as $a[i]$, and $(k[D^\omega X], \cdot, D)$ as $k[D^\omega X]$ or $k\{X\}$. Then $k\{X\}$ has a k -basis as the set (also denote as $[D^\omega X]$) of all words of the form

$$w = a_n[i_n]a_{n-1}[i_{n-1}] \cdots a_1[i_1] \text{ or } w = 1,$$

where $a_t \in X$, $i_t \geq -1$, $1 \leq t \leq n$, $n \in \mathbb{N}$ and $(i_n, a_n) \geq (i_{n-1}, a_{n-1}) \geq \cdots \geq (i_1, a_1)$ lexicographically. For such $w \neq 1$, we define the weight of w , denoted by $wt(w)$, to be $wt(w) = i_1 + i_2 + \cdots + i_n$; the length of w , denoted by $|w|$, to be $|w| = n$; and the $D \cup X$ -length of w , denoted by $|w|_{D \cup X}$, to be $|w|_{D \cup X} = wt(w) + 2n$, which is exactly the number of D and generators from X that occur in w . For $w = 1$, define $wt(w) = |w| = |w|_{D \cup X} = 0$. Furthermore, if we define \circ as

$$f \circ g = (Df)g, \quad f, g \in k\{X\},$$

then $(k\{X\}, \circ)$ becomes a right-Gelfand-Dorfman-Novikov algebra. Its subspace

$$\text{span}_k\{w \in [D^\omega X] \mid wt(w) = -1\},$$

is a subalgebra of $(k\{X\}, \circ)$ (as Gelfand-Dorfman-Novikov algebra), we denote such subalgebra as $GDN_{-1}(X)$. In [16], the authors showed that the Gelfand-Dorfman-Novikov algebra homomorphism $\varphi : GDN(X) \longrightarrow GDN_{-1}(X)$, induced by $\varphi(a) = a[-1]$, is an isomorphism. Therefore, $GDN_{-1}(X)$ is a free Gelfand-Dorfman-Novikov algebra generated by X , which has a k -basis $\{w \in [D^\omega X] \mid wt(w) = -1\}$. From now on, when no ambiguity arises, we denote both $GDN(X)$ and $GDN_{-1}(X)$ as $GDN(X)$ for convenient.

3 Composition-Diamond lemmas

3.1 Monomial order

We order $[D^\omega X]$ as follows.

For any $a[i], b[j] \in D^\omega X$, define

$$a[i] < b[j] \Leftrightarrow (i, a) < (j, b) \text{ lexicographically.}$$

For any $w = a_n[i_n] \cdots a_1[i_1] \in [D^\omega X]$ with $a_n[i_n] \geq \cdots \geq a_1[i_1]$, define

$$\text{ord}(w) \triangleq (|w|, a_n[i_n], \dots, a_1[i_1]).$$

Then, for any $u, v \in [D^\omega X]$ we define

$$u < v \Leftrightarrow \text{ord}(u) < \text{ord}(v) \text{ lexicographically.}$$

It is clear that this is a well order on $[D^\omega X]$. We will use this order throughout this paper.

For any $f \in k\{X\}$, \bar{f} means the leading word of f . We denote the coefficient of \bar{f} as $LC(f)$.

Lemma 3.1. *Let the order $<$ on $[D^\omega X]$ be as above. Then*

$$(i) \quad u < v \Rightarrow u \cdot w < v \cdot w, \quad \overline{Du} < \overline{Dv} \text{ for any } u, v, w \in [D^\omega X], \quad u \neq 1.$$

$$(ii) \quad u < v \Rightarrow \overline{w \circ u} < \overline{w \circ v}, \quad \overline{u \circ w} < \overline{v \circ w} \text{ for any } u, v, w \in [D^\omega X] \setminus \{1\}.$$

Proof. (i) Noting that \cdot is commutative and associative, it is easy to see that $u < v \Rightarrow u \cdot w < v \cdot w$. For any $w = a_n[i_n] \cdots a_1[i_1] \neq 1$, with $a_n[i_n] \geq \cdots \geq a_1[i_1]$, we have $\text{ord}(\overline{Dw}) = (|w|, a_n[i_n + 1], \dots, a_1[i_1])$, so $u < v \Rightarrow \overline{Du} < \overline{Dv}$.

(ii) For any $u, v, w \in [D^\omega X] \setminus \{1\}$, we have $\overline{u \circ w} = \overline{(Du)w} = \overline{Du} \cdot w$ and $\overline{v \circ w} = \overline{(Dv)w} = \overline{Dv} \cdot w$, so $u < v \Rightarrow \overline{u \circ w} < \overline{v \circ w}$. By the same reasoning, $u < v \Rightarrow \overline{w \circ u} < \overline{w \circ v}$. ■

3.2 S -words

For any $S \subseteq k\{X\}$, we denote $Id[S]$ the ideal of $k\{X\}$ generated by S and

$$k\{X|S\} \triangleq k\{X\}/Id[S]$$

the commutative differential algebra generated by X with defining relations S . Since $Id[S]$ is closed under \cdot and the derivation D , we have

$$Id[S] = \text{span}_k\{uD^t s \mid u \in [D^\omega X], t \in \mathbb{N}, s \in S\}.$$

For any $u \in [D^\omega X], t \in \mathbb{N}, s \in S$, we call $uD^t s$ an S -word in $k\{X\}$. We call $uD^t s$ an S -word in $GDN(X)$ if $wt(\overline{uD^t s}) = -1$ and $S \subseteq GDN(X)$.

Suppose $S \subseteq GDN(X)$ and denote $Id(S)$ the ideal of $GDN(X)$ generated by S . Then we have the following lemma.

Lemma 3.2. *Suppose $S \subseteq GDN(X)$. Then*

$$Id(S) = span_k\{uD^t s \mid u \in [D^\omega X], t \in \mathbb{N}, s \in S, wt(\overline{uD^t s}) = -1\}.$$

Proof. It is clear that the right part is an ideal that contains S . We just need to show that $uD^t s \in Id(S)$ whenever $wt(\overline{uD^t s}) = -1$. Since $wt(\overline{uD^t s}) = -1$, we have

$$uD^t s = c_1[i_1] \cdots c_n[i_n] a_1[-1] \cdots a_m[-1] (D^t s) b_1[-1] \cdots b_t[-1],$$

where $u = c_1[i_1] \cdots c_n[i_n] a_1[-1] \cdots a_m[-1] b_1[-1] \cdots b_t[-1]$, $m = i_1 + \cdots + i_n$ and $i_n \geq i_{n-1} \geq \cdots \geq i_1 \geq 0$. So the lemma will be clear if we show

$$(i) \quad (D^t s) b_1[-1] \cdots b_t[-1] \in Id(S) \text{ whenever } s \in S;$$

$$(ii) \quad c[p] a_1[-1] \cdots a_p[-1] f \in Id(S) \text{ whenever } f \in Id(S).$$

To prove (i), we use induction on t . If $t = 0$, it is clear. Suppose that it holds for all $t \leq n$. Then

$$\begin{aligned} & (D^{n+1} s) b_1[-1] \cdots b_{n+1}[-1] \\ &= ((D^n s) b_1[-1] \cdots b_n[-1]) \circ b_{n+1}[-1] \\ &\quad - \sum_{1 \leq i \leq n} (D^n s) b_1[-1] \cdots (D b_i[-1]) \cdots b_n[-1] \cdot b_{n+1}[-1] \\ &= ((D^n s) b_1[-1] \cdots b_n[-1]) \circ b_{n+1}[-1] \\ &\quad - \sum_{1 \leq i \leq n} b_i[-1] \circ ((D^n s) b_1[-1] \cdots b_{i-1}[-1] b_{i+1}[-1] \cdots b_{n+1}[-1]) \\ &\in Id(S). \end{aligned}$$

To prove (ii), we use induction on p . If $p = 0$, it is clear. Suppose that it holds for all $p \leq n$. Then

$$\begin{aligned} & c[n+1] a_1[-1] \cdots a_{n+1}[-1] f \\ &= (c[n] a_1[-1] \cdots a_{n+1}[-1]) \circ f \\ &\quad - \sum_{1 \leq i \leq n+1} c[n] a_1[-1] \cdots (D a_i[-1]) \cdots a_{n+1}[-1] \cdot f \\ &= (c[n] a_1[-1] \cdots a_{n+1}[-1]) \circ f \\ &\quad - \sum_{1 \leq i \leq n+1} c[n] a_1[-1] \cdots a_{i-1}[-1] a_{i+1}[-1] \cdots a_{n+1}[-1] \cdot (a_i[-1] \circ f) \\ &\in Id(S). \end{aligned}$$

So $Id(S) = span_k\{uD^t s \mid u \in [D^\omega X], t \in \mathbb{N}, s \in S, wt(\overline{uD^t s}) = -1\}$. ■

Let S be a subset of $k\{X\}$. We call S homogeneous (weight homogeneous, $D \cup X$ -homogeneous, resp.), if for any $f = \sum_{j=1}^q \beta_j w_j \in S$, we have $|w_1| = \dots = |w_q|$ ($wt(w_1) = \dots = wt(w_q)$, $|w_1|_{D \cup X} = \dots = |w_q|_{D \cup X}$, resp.). We have the following lemma immediately.

Lemma 3.3. *Let $S \subseteq k\{X\}$, $f = \sum_{i \in I} \beta_i u_i D^{t_i} s_i$, where each $\beta_i \in k, u_i \in [D^\omega X], s_i \in S, t_i \in \mathbb{N}$. If f and S are homogeneous (weight homogeneous, $D \cup X$ -homogeneous, resp.), then we can suppose that $|\overline{u_i D^{t_i} s_i}| = |\overline{f}|$ ($wt(\overline{u_i D^{t_i} s_i}) = wt(\overline{f})$, $|\overline{u_i D^{t_i} s_i}|_{D \cup X} = |\overline{f}|_{D \cup X}$, resp.) for any $i \in I$.*

3.3 Composition-Diamond lemma for commutative differential algebras

The idea of this subsection is essentially the same as the construction of standard differential Gröbner bases in [17, 27], in which the authors deal with more general case with several derivations.

For any $u, v \in [D^\omega X]$, we always denote $lcm(u, v)$ the least common multiple of u, v in $[D^\omega X]$. We call $lcm(u, v)$ a non-trivial least common multiple of u and v if $|lcm(u, v)| < |uv|$.

For any $f, g \in S \subseteq k\{X\}$, if $w = lcm(\overline{D^{t_1} f}, \overline{D^{t_2} g})$ is a non-trivial least common multiple of $\overline{D^{t_1} f}$ and $\overline{D^{t_2} g}$, then we call

$$[D^{t_1} f, D^{t_2} g]_w = \frac{1}{\alpha_1} w|_{\overline{D^{t_1} f} \mapsto D^{t_1} f} - \frac{1}{\alpha_2} w|_{\overline{D^{t_2} g} \mapsto D^{t_2} g}$$

a composition for $D^{t_1} f \wedge D^{t_2} g$ corresponding to w , where $\alpha_1 = LC(D^{t_1} f), \alpha_2 = LC(D^{t_2} g)$.

For a polynomial $h \in k\{X\}$, we say $h \equiv 0 \mod(S, w)$ if $h = \sum \gamma_i u_i D^{t_i} s_i$, where each $\gamma_i \in k, u_i D^{t_i} s_i$ is an S -word and $\overline{u_i D^{t_i} s_i} < w$. Denote $h \equiv h' \mod(S, w)$ if $h - h' \equiv 0 \mod(S, w)$. The composition $[D^{t_1} f, D^{t_2} g]_w$ is trivial $\mod(S, w)$ if $[D^{t_1} f, D^{t_2} g]_w \equiv 0 \mod(S, w)$.

For $f, g \in S, w = lcm(\overline{D^{t_1} f}, \overline{D^{t_2} g})$, if $w = \overline{f}$ or $w = \overline{g}$, then the composition is called inclusion; Otherwise, the composition is called intersection.

Definition 3.1. Let S be a non-empty subset of $k\{X\}$. Then the set S is called a Gröbner-Shirshov basis in $k\{X\}$ if all compositions of S in $k\{X\}$ are trivial.

Theorem 1. (Composition-Diamond lemma for commutative differential algebras) [17, 27] Let $<$ be the monomial order on $k\{X\}$ as before and S a non-empty subset of $k\{X\}$. Let $Id[S]$ be the ideal of $k\{X\}$ generated by S . Then the following statements are equivalent.

- (i) S is a Gröbner-Shirshov basis in $k\{X\}$.
- (ii) $0 \neq h \in Id[S] \Rightarrow \bar{h} = u\overline{D^t s}$ for some $s \in S, u \in [D^\omega X], t \in \mathbb{N}$.
- (iii) $Irr[S] = \{w \in [D^\omega X] \mid w \neq u\overline{D^t s}, u \in [D^\omega X], t \in \mathbb{N}, s \in S\}$ is a linear basis for $k\{X|S\}$.

Buchberger-Shirshov algorithm: If a subset $S \subset k\{X\}$ is not a Gröbner-Shirshov basis then one can add all non-trivial compositions of S to S . Continuing this process repeatedly, we finally obtain a Gröbner-Shirshov basis S^c that contains S . Such a process is called Buchberger-Shirshov algorithm.

3.4 Composition-Diamond lemma for Gelfand-Dorfman-Novikov algebras

For any $u, v, w \in [D^\omega X]$, we call w a common multiple of u and v in $GDN(X)$ if $wt(w) = -1$ and w is a common multiple of u and v in $[D^\omega X]$; w is a non-trivial common multiple of u and v in $GDN(X)$ if w is a common multiple of u and v in $GDN(X)$ such that $w \neq uvw'$ for any $w' \in [D^\omega X]$.

Let $f, g \in GDN(X)$ and w a non-trivial common multiple of $\overline{D^{t_1} f}$ and $\overline{D^{t_2} g}$ in $GDN(X)$. Then a composition of $D^{t_1} f \wedge D^{t_2} g$ relative to w is defined as

$$(D^{t_1} f, D^{t_2} g)_w = \frac{1}{\alpha_1} w|_{\overline{D^{t_1} f} \rightarrow D^{t_1} f} - \frac{1}{\alpha_2} w|_{\overline{D^{t_2} g} \rightarrow D^{t_2} g},$$

where $\alpha_1 = LC(D^{t_1} f)$ and $\alpha_2 = LC(D^{t_2} g)$.

Suppose that $S \subseteq GDN(X)$ and $h \in GDN(X)$. Then we say $h \equiv 0 \pmod{(S, w)}$ if $h = \sum \beta_i u_i D^{t_i} s_i$, where each $\beta_i \in k$, $u_i D^{t_i} s_i$ is an S -word such that $wt(\overline{u_i D^{t_i} s_i}) = -1$ and $\overline{u_i D^{t_i} s_i} < w$. The composition $(D^{t_1} f, D^{t_2} g)_w$ is trivial $\pmod{(S, w)}$ if $(D^{t_1} f, D^{t_2} g)_w \equiv 0 \pmod{(S, w)}$.

Let

$$w = lcm(\overline{D^{t_1} f}, \overline{D^{t_2} g}) d_1[m_1] \cdots d_p[m_p] c_1[-1] \cdots c_q[-1]$$

be a non-trivial common multiple of $\overline{D^{t_1} f}$ and $\overline{D^{t_2} g}$ in $GDN(X)$, where $d_1[m_1] \geq \cdots \geq d_p[m_p]$, $m_p > 0$, $c_1 \geq \cdots \geq c_q$. Then we say w to be critical if one of the following holds:

- (i) If $wt(lcm(\overline{D^{t_1} f}, \overline{D^{t_2} g})) > -1$, then $d_1[m_1] \cdots d_p[m_p]$ is empty.
- (ii) If $wt(lcm(\overline{D^{t_1} f}, \overline{D^{t_2} g})) = -1$, then $d_1[m_1] \cdots d_p[m_p] c_1[-1] \cdots c_q[-1]$ is empty.
- (iii) If $wt(lcm(\overline{D^{t_1} f}, \overline{D^{t_2} g})) < -1$, then $wt(lcm(\overline{D^{t_1} f}, \overline{D^{t_2} g}) d_1[m_1] \cdots d_{p-1}[m_{p-1}]) < -1$ and $wt(lcm(\overline{D^{t_1} f}, \overline{D^{t_2} g}) d_1[m_1] \cdots d_p[m_p]) \geq -1$.

Definition 3.2. Let S be a non-empty subset of $GDN(X)$. Then the set S is called a Gröbner-Shirshov basis in $GDN(X)$ if all compositions of S in $GDN(X)$ are trivial.

Lemma 3.4. Suppose that the composition $(D^{t_1} f, D^{t_2} g)_w$ is trivial for every critical common multiple w of $\overline{D^{t_1} f}$ and $\overline{D^{t_2} g}$, where $f, g \in S$, $t_1, t_2 \in \mathbb{N}$. Then S is a Gröbner-Shirshov basis in $GDN(X)$.

Proof. Noting that any common multiple of $\overline{D^{t_1} f}$ and $\overline{D^{t_2} g}$ in $GDN(X)$ contains some critical common multiple w of $\overline{D^{t_1} f}$ and $\overline{D^{t_2} g}$, the result follows. \blacksquare

Lemma 3.5. Suppose that S is a Gröbner-Shirshov basis in $GDN(X)$, $f, g \in S$ and $w = \overline{u D^t f} = \overline{v D^{t'} g} \in [D^\omega X]$, $wt(w) = -1$. Then $\frac{1}{\alpha_1} u D^t f - \frac{1}{\alpha_2} v D^{t'} g \equiv 0 \pmod{(S, w)}$, where $\alpha_1 = LC(u D^t f)$ and $\alpha_2 = LC(v D^{t'} g)$.

Proof. If $u = u' \overline{D^{t'} g}$ for some $u' \in [D^\omega X]$, then $v = u' \overline{D^t f}$. Thus

$$\begin{aligned} & \frac{1}{\alpha_1} u D^t f - \frac{1}{\alpha_2} v D^{t'} g \\ &= \frac{1}{\alpha_1} u' (\overline{D^{t'} g}) D^t f - \frac{1}{\alpha_1 \alpha_2} u' (D^t f) D^{t'} g + \frac{1}{\alpha_1 \alpha_2} u' (D^t f) D^{t'} g - \frac{1}{\alpha_2} u' (\overline{D^t f}) D^{t'} g \\ &= \frac{1}{\alpha_1} (\overline{D^{t'} g} - \frac{1}{\alpha_2} D^{t'} g) u' D^t f - \frac{1}{\alpha_2} (\overline{D^t f} - \frac{1}{\alpha_1} D^t f) u' D^{t'} g \\ &\equiv 0 \pmod{(S, w)}. \end{aligned}$$

Otherwise, w is a non-trivial common multiple of $\overline{D^t f}$ and $\overline{D^{t'} g}$ in $GDN(X)$. Since S is a Gröbner-Shirshov basis, by definition we have $\frac{1}{\alpha_1} u D^t f - \frac{1}{\alpha_2} v D^{t'} g \equiv 0 \pmod{(S, w)}$. ■

Lemma 3.6. *Let S be a non-empty subset of $GDN(X)$. Denote*

$$Irr(S) = \{w \in [D^\omega X] \mid w \neq \overline{u D^t s}, u \in [D^\omega X], t \in \mathbb{N}, s \in S, wt(w) = -1\}.$$

Then for all $h \in GDN(X)$, we have

$$h = \sum_{u_i \overline{D^{t_i} s_i} \leq \overline{h}} \beta_i u_i D^{t_i} s_i + \sum_{w_j \leq \overline{h}} \gamma_j w_j,$$

where each $\beta_i, \gamma_j \in k$, $u_i \in [D^\omega X]$, $t_i \in \mathbb{N}$, $s_i \in S$, $w_j \in Irr(S)$, and $wt(u_i \overline{D^{t_i} s_i}) = wt(w_j) = -1$.

Proof. By induction on \overline{h} , we have the result. ■

Theorem 2. *(Composition-Diamond lemma for Gelfand-Dorfman-Novikov algebras) Let S be a non-empty subset of $GDN(X)$ and $Id(S)$ be the ideal of $GDN(X)$ generated by S . Then the following statements are equivalent.*

(i) S is a Gröbner-Shirshov basis in $GDN(X)$.

(ii) $0 \neq h \in Id(S) \Rightarrow \overline{h} = u \overline{D^t s}$ for some $s \in S, u \in [D^\omega X], t \in \mathbb{N}$.

(iii) $Irr(S) = \{w \in [D^\omega X] \mid w \neq \overline{u D^t s}, u \in [D^\omega X], t \in \mathbb{N}, s \in S, wt(w) = -1\}$ is a linear basis for $GDN(X|S) \triangleq GDN(X)/Id(S)$.

Proof. (i) \Rightarrow (ii). Let S be a Gröbner-Shirshov basis and $0 \neq h \in Id(S)$. Then h has an expression $h = \sum_{i=1}^n \beta_i u_i D^{t_i} s_i$, where each $0 \neq \beta_i \in k$, $u_i \in [D^\omega X]$, $t_i \in \mathbb{N}$, $s_i \in S$, $wt(u_i \overline{D^{t_i} s_i}) = -1$. Denote $w_i = u_i \overline{D^{t_i} s_i}$, $i = 1, 2, \dots, n$. We may assume without loss of generality that

$$w_1 = w_2 = \dots = w_l > w_{l+1} \geq w_{l+2} \geq \dots$$

for some $l \geq 1$. Then $w_1 \geq \overline{h}$.

We show the result by induction on (w_1, l) , where for any $l, l' \in \mathbb{N}$ and $w, w' \in [D^\omega X]$, $(w, l) < (w', l')$ lexicographically. We call (w_1, l) the height of h .

If $\overline{h} = w_1$ or $l = 1$, then the result is obvious.

Now suppose that $w_1 > \bar{h}$. Then $l > 1$ and $u_1 \overline{D^{t_1} s_1} = u_2 \overline{D^{t_2} s_2}$. By Lemma 3.5, we have

$$\begin{aligned} & \beta_1 u_1 D^{t_1} s_1 + \beta_2 u_2 D^{t_2} s_2 \\ = & \beta_1 (u_1 D^{t_1} s_1 - \frac{\alpha_1}{\alpha_2} u_2 D^{t_2} s_2) + \frac{\alpha_1 \beta_1 + \alpha_2 \beta_2}{\alpha_2} u_2 D^{t_2} s_2 \\ \equiv & \frac{\alpha_1 \beta_1 + \alpha_2 \beta_2}{\alpha_2} u_2 D^{t_2} s_2 \pmod{(S, w_1)}, \end{aligned}$$

where $\alpha_i = LC(u_i D^{t_i} s_i)$, $i = 1, 2$. Thus,

$$h = \frac{\alpha_1 \beta_1 + \alpha_2 \beta_2}{\alpha_2} u_2 D^{t_2} s_2 + \sum_{i=3}^n \beta_i u_i D^{t_i} s_i + \sum_{j=1}^m \gamma_j v_j D^{t'_j} s'_j, \quad v_j \overline{D^{t'_j} s'_j} < w_1,$$

which has height $< (w_1, l)$. Now the result follows by induction.

(ii) \Rightarrow (iii). By Lemma 3.6, the set $Irr(S)$ generates the algebra $GDN(X|S)$ as a k -vector space. On the other hand, suppose that $\sum_{1 \leq i \leq n} \gamma_i w_i = 0$ in $GDN(X|S)$, where each $0 \neq \gamma_i \in k$, $w_i \in Irr(S)$ and $w_1 > w_2 > \dots > w_n$. Then we have $\sum_{1 \leq i \leq n} \gamma_i w_i = \sum_{1 \leq j \leq m} \beta_j u_j D^{t_j} s_j \neq 0$ in $GDN(X)$. So by (ii) we get $w_1 \notin Irr(S)$, which contradicts to the choice of w_1 .

(iii) \Rightarrow (i). For any $f, g \in S$, $t_1, t_2 \in \mathbb{N}$, denote w a non-trivial common multiple of $\overline{D^{t_1} f}$ and $\overline{D^{t_2} g}$. Then by Lemma 3.6, we have

$$(D^{t_1} f, D^{t_2} g)_w = \sum_{u_i \overline{D^{t_i} s_i} < w} \beta_i u_i D^{t_i} s_i + \sum_{w_i < w} \gamma_j w_j,$$

where each $\beta_i, \gamma_j \in k$, $u_i \in [D^\omega X]$, $t_i \in \mathbb{N}$, $w_j \in Irr(S)$ and $wt(u_i \overline{D^{t_i} s_i}) = wt(w_j) = -1$. Since $(D^{t_1} f, D^{t_2} g)_w \in Id(S)$ and by (iii), we have

$$(D^{t_1} f, D^{t_2} g)_w \equiv 0 \pmod{(S, w)}.$$

Therefore, S is a Gröbner-Shirshov basis in $GDN(X)$. ■

Since for any Gelfand-Dorfman-Novikov tableau

$$w = Y_p \circ (Y_{p-1} \circ (\dots \circ (Y_2 \circ Y_1) \dots)),$$

we have

$$\overline{w} = \overline{DY_p} \cdot \overline{DY_{p-1}} \cdots \overline{DY_2} \cdot \overline{Y_1}$$

and

$$\begin{aligned}\overline{DY_i} &= a_{i,1}[r_i - 1]a_{i,r_i}[-1] \cdots a_{i,3}[-1]a_{i,2}[-1], \quad 2 \leq i \leq p, \\ \overline{Y_1} &= a_{1,1}[r_1 - 1]a_{1,r_1+1}[-1]a_{1,r_1}[-1] \cdots a_{1,3}[-1]a_{1,2}[-1],\end{aligned}$$

where

$$\begin{aligned}Y_i &= (\cdots((a_{i,1} \circ a_{i,2}) \circ a_{i,3}) \cdots \circ a_{i,r_i}), \quad 2 \leq i \leq p, \\ Y_1 &= (\cdots((a_{1,1} \circ a_{1,2}) \circ a_{1,3}) \cdots \circ a_{1,r_1}) \circ a_{1,r_1+1},\end{aligned}$$

we immediately get the following proposition:

Proposition 3.1. *If $S \subseteq \text{GDN}(X)$ is a Gröbner-Shirshov basis, then the set $\{w \in \text{GDN}(X) \mid w \text{ is a Gelfand-Dorfman-Novikov tableau, } \overline{w} \in \text{Irr}(S)\}$ is a linear basis for $\text{GDN}(X|S)$.*

4 Applications

4.1 An example

In the paper [12], the authors list a lot of left-Gelfand-Dorfman-Novikov algebras in low dimensions. We can get their corresponding right-Gelfand-Dorfman-Novikov algebras using $a \circ_{op} b \triangleq b \circ a$, see also [15].

Example 4.1. ([12]) *Let $X = \{e_1, e_2, e_3, e_4\}$, $S = \{e_2[0]e_1[-1] = e_3[-1], e_3[0]e_1[-1] = e_4[-1], e_i[0]e_j[-1] = 0, \text{ if } (i, j) \notin \{(2, 1), (3, 1)\}, 1 \leq i, j \leq 4\}$. Then S is a Gröbner-Shirshov basis in $\text{GDN}(X)$. It follows from Theorem 2 that $\{e_1[-1], e_2[-1], e_3[-1], e_4[-1]\}$ is a linear basis of the Gelfand-Dorfman-Novikov algebra $\text{GDN}(X|S)$.*

Proof. Denote

$$f_{ij} : e_i[0]e_j[-1] = \sum_{1 \leq l \leq 4} \alpha_{ij}^l e_l[-1] \in S, \quad 1 \leq i, j \leq 4.$$

Before checking the compositions, we prove the following claims.

Claim (i): Let $w = e_i[n]e_{i_1}[-1] \cdots e_{i_{n+1}}[-1]$, $n \geq 0$. Then $w = \sum \alpha_j u_j D^{t_j} s_j$, where each $\overline{u_j D^{t_j} s_j} \leq w$, $s_j \in S$ if $i_l \neq 1$ for some $1 \leq l \leq n+1$.

We show Claim (i) by induction on n . For $n = 0$ or 1 , the result follows immediately. Suppose $t \geq 2$ and the result holds for any $n < t$. Then

$$\begin{aligned}
w &= e_i[t]e_{i_1}[-1] \cdots e_{i_{t+1}}[-1] \\
&= D^t(e_i[0]e_{i_1}[-1] - \sum_{1 \leq m \leq 4} \alpha_{i,i_1}^m e_m[-1])e_{i_2}[-1] \cdots e_{i_{t+1}}[-1] \\
&\quad - \sum_{0 \leq p \leq t-1} \binom{t}{p} e_i[p]e_{i_1}[t-1-p]e_{i_2}[-1] \cdots e_{i_{t+1}}[-1] \\
&\quad + \sum_{1 \leq m \leq 4} \alpha_{i,i_1}^m e_m[t-1]e_{i_2}[-1] \cdots e_{i_{t+1}}[-1].
\end{aligned}$$

If for any $1 \leq l \leq n+1$, $i_l \neq 1$, then by induction hypothesis, the result follows immediately. Otherwise, say $i_1 = 1$, then $i_l \neq 1$ for some $2 \leq l \leq n+1$. By induction hypothesis, the result follows immediately.

Claim (ii): For any $n_1, n_2 \geq 0$, we have $w = e_l[n_1]e_i[n_2]e_{i_1}[-1] \cdots e_{i_{n_1+n_2+1}}[-1] = \sum \alpha_j u_j D^{t_j} s_j$, with each $\overline{u_j D^{t_j} s_j} \leq w$.

We show Claim (ii) by induction on n_1 . If $n_1 = 0$, then

$$\begin{aligned}
w &= (e_l[0]e_{i_1}[-1] - \sum_{1 \leq m \leq 4} \alpha_{l,i_1}^m e_m[-1])e_i[n_2]e_{i_2}[-1] \cdots e_{i_{n_1+n_2+1}}[-1] \\
&\quad + \sum_{1 \leq m \leq 4} \alpha_{l,i_1}^m e_m[-1]e_i[n_2]e_{i_2}[-1] \cdots e_{i_{n_1+n_2+1}}[-1].
\end{aligned}$$

By Claim (i), the result follows immediately. If $n_1 > 0$, then

$$\begin{aligned}
w &= D^{n_1}(e_l[0]e_{i_1}[-1] - \sum_{1 \leq m \leq 4} \alpha_{l,i_1}^m e_m[-1])e_i[n_2]e_{i_2}[-1] \cdots e_{i_{n_1+n_2+1}}[-1] \\
&\quad - \sum_{0 \leq p \leq n_1-1} \binom{n_1}{p} e_l[p]e_{i_1}[n_1-1-p]e_i[n_2]e_{i_2}[-1] \cdots e_{i_{n_1+n_2+1}}[-1] \\
&\quad + \sum_{1 \leq m \leq 4} \alpha_{l,i_1}^m e_m[n_1-1]e_i[n_2]e_{i_2}[-1] \cdots e_{i_{n_1+n_2+1}}[-1].
\end{aligned}$$

By induction hypothesis, the result follows immediately.

For any $t \in \mathbb{N}$, $u \in [D^\omega X]$, if $wt(\overline{(D^t f_{ij})u}) = -1$, $|u| > 0$ and $\overline{(D^t f_{ij})u} \neq e_i[t](e_1[-1])^{t+1}$, then by Claims (i) and (ii), we have

$$(D^t f_{ij})u - \overline{(D^t f_{ij})u} = \sum_{0 \leq p \leq t-1} e_i[p]e_j[t-1-p]u + \sum_{1 \leq m \leq 4} \alpha_{i,j}^m e_m[t-1]u \equiv 0 \pmod{(S, \overline{(D^t f_{ij})u})}.$$

Since for any $t \in \mathbb{N}$, $j \neq l$, each critical common multiple of $D^t f_{ij} \wedge D^t f_{il}$ has form $w = e_i[t]e_j[-1]e_l[-1]e_{i_1}[-1] \cdots e_{i_{t-1}}[-1]$, we get

$$(D^t f_{ij}, D^t f_{il})_w \equiv w - w \equiv 0 \pmod{(S, w)}.$$

For the case of $D^{t_1} f_{i_1 j} \wedge D^{t_2} f_{i_2 j}$, where $t_1 \neq t_2$ or $i_1 \neq i_2$, the proof is almost the same. So S is a Gröbner-Shirshov basis in $GDN(X)$. ■

4.2 PBW type theorem in Shirshov form

Theorem 3. (*PBW type theorem in Shirshov form*) Let $GDN(X)$ be a free Gelfand-Dorfman-Novikov algebra, $k\{X\}$ be a free commutative differential algebra, $S \subseteq GDN(X)$ and S^c a Gröbner-Shirshov basis in $k\{X\}$, which is obtained from S by Buchberger-Shirshov algorithm. Then

(i) $S' = \{uD^m s \mid s \in S^c, u \in [D^\omega X], m \in \mathbb{N}, wt(u\overline{D^m s}) = -1\}$ is a Gröbner-Shirshov basis in $GDN(X)$.

(ii) The set $Irr(S') = \{w \in [D^\omega X] \mid w \neq u\overline{D^t s}, u \in [D^\omega X], t \in \mathbb{N}, s \in S^c, wt(w) = -1\} = GDN(X) \cap Irr[S^c]$ is a linear basis of $GDN(X|S)$. Thus, any Gelfand-Dorfman-Novikov algebra $GDN(X|S)$ is embeddable into its universal enveloping commutative differential algebra $k\{X|S\}$.

Proof. (i). We first show that any $h \in S^c$ has the form $h = \sum_{i \in I_h} \gamma_i w_i$, with each $\gamma_i \neq 0$, $wt(w_i) = wt(w_{i'})$, $i, i' \in I_h$. Suppose

$$f = \sum_{i \in I_f} \beta_i w_i, \quad \text{with } wt(w_i) = wt(w_{i'}) \text{ for any } i, i' \in I_f,$$

$$g = \sum_{i \in I_g} \beta_i w_i, \quad \text{with } wt(w_i) = wt(w_{i'}), \text{ for any } i, i' \in I_g,$$

and

$$(D^t f, D^{t'} g)_{w'} = \frac{1}{\alpha_1} u D^t f - \frac{1}{\alpha_2} v D^{t'} g = \sum_{j \in J} \gamma_j w_j \text{ in } k\{X\}.$$

Then it is obvious that $wt(w_j) = wt(w_{j'}), \forall j, j' \in J$. So whenever we add some non-trivial composition to S while doing the Buchberger-Shirshov algorithm, any monomial of such a composition will share the same weight. It follows that $S' \subseteq GDN(X)$.

If $w = w_1 \overline{D^{t_1}(u_1 D^{m_1} s_1)} = w_2 \overline{D^{t_2}(u_2 D^{m_2} s_2)} \in GDN(X)$ is a non-trivial common multiple of $\overline{D^{t_1}(u_1 D^{m_1} s_1)}$ and $\overline{D^{t_2}(u_2 D^{m_2} s_2)}$, where $s_1, s_2 \in S^c$, $t_1, t_2 \in \mathbb{N}$, $f = u_1 D^{m_1} s_1$, $g = u_2 D^{m_2} s_2 \in S'$, then by Theorem 1, we have

$$(D^{t_1} f, D^{t_2} g)_w = \frac{1}{\alpha_1} w_1 D^{t_1}(u_1 D^{m_1} s_1) - \frac{1}{\alpha_2} w_2 D^{t_2}(u_2 D^{m_2} s_2) = \sum_{l \in L} \delta_l u_l D^{j_l} s_l,$$

where each $\delta_l \in k$, $u_l \in [D^\omega X]$, $s_l \in S^c$, $j_l \in \mathbb{N}$, $\overline{u_l D^{j_l} s_l} < w$. Furthermore, by Lemma 3.3, we can assume that for each $l \in L$, $wt(\overline{u_l D^{j_l} s_l}) = -1$, which means $(D^{t_1} f, D^{t_2} g)_w \equiv 0 \pmod{(S', w)}$. So S' is a Gröbner-Shirshov basis in $GDN(X)$.

It remains to show that the ideal $Id(S)$ of $GDN(X)$ generated by S is $Id(S')$. It is clear that $S \subseteq Id(S')$. Since $S^c \subseteq Id[S^c] = Id[S]$, for any $s \in S^c$, we have $s = \sum \beta_i u_i D^{t_i} s_i$, where each $\beta_i \in k$, $u_i \in [D^\omega X]$, $s_i \in S$ and $wt(\overline{u_i D^{t_i} s_i}) = wt(\overline{s})$. By Lemma 3.2, it follows that $S' \subseteq Id(S)$.

(ii). Since $\{w \in [D^\omega X] \mid w \neq \overline{u D^t s}, u \in [D^\omega X], t \in \mathbb{N}, s \in S', wt(w) = -1\} = \{w \in [D^\omega X] \mid w \neq \overline{u D^t s}, u \in [D^\omega X], t \in \mathbb{N}, s \in S^c, wt(w) = -1\}$ by (i), we have $Irr(S') \subseteq Irr[S^c]$. The result follows immediately. \blacksquare

Remark 4.1. Theorem 3 essentially offers another way to calculate Gröbner-Shirshov basis in $GDN(X)$ and it indicates some close connection between $GDN(X|S)$ and its universal enveloping algebra $k\{X|S\}$. In fact, by Lemma 3.2, we have $GDN(X) \cap Id[S] = Id(S)$. It is clear that $Id[S]$ is a subalgebra of $(k\{X\}, \circ)$ as Gelfand-Dorfman-Novikov algebra. Then we have a Gelfand-Dorfman-Novikov algebra isomorphism as follows:

$$GDN(X)/Id(S) = GDN(X)/(Id[S] \cap GDN(X)) \cong (GDN(X) + Id[S])/Id[S] \leq (k\{X|S\}, \circ).$$

4.3 Algorithms for word problems

The general observation shows that for a homogeneous variety the word problem in an algebra with finite number of homogeneous relations is always algorithmically solvable. In this subsection, we will provide algorithms for solving such word problems.

Let $k\{X|S\}$ be a commutative differential algebra and $S = \{f_i \mid 1 \leq i \leq p\}$, $p \in \mathbb{N}$, where S is $D \cup X$ -homogeneous in the sense that for any $f = \sum_{j=1}^q \beta_j w_j \in S$, we have $|w_1|_{D \cup X} = |w_2|_{D \cup X} = \dots = |w_q|_{D \cup X}$.

In this subsection, we always assume that $S \subset k\{X\}$ is a non-empty $D \cup X$ -homogeneous set. We call S a minimal set, if there are no $f, g \in S$ with $f \neq g$, such that $\bar{f} = u\overline{D^t g}$ for any $u \in [D^\omega X], t \in \mathbb{N}$. For any $f, g \in S$, if $\bar{f} = u\overline{D^t g}$ and the composition $[f, D^t g]_{\bar{f}} = \frac{1}{\alpha_1}f - \frac{1}{\alpha_2}u D^t g \equiv 0 \pmod{(S, w)}$, then we delete f from S to reduce the set S in one step to a new set S_0 , i.e., $S \longrightarrow S_0 = S \setminus \{f\}$; If $\bar{f} = u\overline{D^t g}$ and the composition $[f, D^t g]_{\bar{f}} = \frac{1}{\alpha_1}f - \frac{1}{\alpha_2}u D^t g \not\equiv 0 \pmod{(S, w)}$, then we replace f by $h \triangleq \frac{1}{\alpha_1}f - \frac{1}{\alpha_2}u D^t g$ to reduce the set S in one step to a new set S_0 , i.e., $S \longrightarrow S_0 = (S \setminus \{f\}) \cup \{h\}$, where $\alpha_1 = LC(f)$ and $\alpha_2 = LC(D^t g)$. In both cases, we say that f is reduced by g . It is clear that S_0 is also a $D \cup X$ -homogeneous set.

Lemma 4.1. *If $|S| < \infty$ and S is $D \cup X$ -homogeneous, then we can effectively reduce S into a minimal $D \cup X$ -homogeneous set $S^{(0)}$ in finitely many steps, such that $Id[S] = Id[S^{(0)}]$ and for any $f \in S$, we have $f = \sum \beta_q u_q D^{t_q} s_q$, with $|\overline{u_q D^{t_q} s_q}|_{D \cup X} = |\bar{f}|_{D \cup X}$ and $\overline{u_q D^{t_q} s_q} \leq \bar{f}$, where each $\beta_q \in k, u_q \in [D^\omega X], s_q \in S^{(0)}, t_q \in \mathbb{N}$.*

Proof. Suppose $S = \{f_i \mid 1 \leq i \leq p\}, p \in \mathbb{N}$ and $1 \leq |\bar{f}_1|_{D \cup X} \leq |\bar{f}_2|_{D \cup X} \leq \dots \leq |\bar{f}_p|_{D \cup X}$. Given $f, g \in S$, suppose

$$\bar{f} = a_n[i_n] \cdots a_1[i_1] \text{ and } \bar{g} = b_m[j_m] \cdots b_1[j_1],$$

with $a_n[i_n] \geq \dots \geq a_1[i_1], b_m[j_m] \geq \dots \geq b_1[j_1]$ and $j_m \leq i_n$. To decide whether g can reduce f or not, we only need to check whether one of $\bar{g}, \overline{D^1 g}, \dots, \overline{D^{i_n - j_m} g}$ is a subword of \bar{f} or not. Define $ord(S) = (p, \overline{f_p}, \overline{f_{p-1}}, \dots, \overline{f_1})$. Then if one reduce S in one step to S_{01} , we have $ord(S_{01}) < ord(S)$ lexicographically. Therefore, S can be reduced into a minimal set $S^{(0)}$ in finitely many steps, say, $S \longrightarrow S_{01} \longrightarrow S_{02} \longrightarrow \dots \longrightarrow S_{0l} = S^{(0)}$. Then by induction on l , we easily get each $Id[S_{0m}] = Id[S]$ and for any $f \in S$, we have $f = \sum \beta_q u_q D^{t_q} s_q$, with $|\overline{u_q D^{t_q} s_q}|_{D \cup X} = |\bar{f}|_{D \cup X}$ and $\overline{u_q D^{t_q} s_q} \leq \bar{f}$, where $1 \leq m \leq l, \beta_q \in k, u_q \in [D^\omega X], s_q \in S_{0m}, t_q \in \mathbb{N}$. ■

Suppose that S is a minimal set and

$$S = S^{(0)} = \{f_i \mid 1 \leq i \leq p\}, p \in \mathbb{N},$$

where $1 \leq |\bar{f}_1|_{D \cup X} \leq |\bar{f}_2|_{D \cup X} \leq \dots \leq |\bar{f}_p|_{D \cup X}$. For any $f, g \in S^{(0)}, t_1, t_2 \in \mathbb{N}, t_1, t_2 \leq 1, w = lcm(\overline{D^{t_1} f}, \overline{D^{t_2} g})$, we will check composition $[D^{t_1} f, D^{t_2} g]_w$ whenever w is a non-trivial common multiple of $\overline{D^{t_1} f}$ and $\overline{D^{t_2} g}$. If all such compositions are trivial, we just

set $S_1 = S^{(0)}$. Otherwise, if for some $t_1, t_2 \leq 1, w = lcm(\overline{D^{t_1}f}, \overline{D^{t_2}g})$, the non-trivial composition $[D^{t_1}f, D^{t_2}g]_w = \sum_{i \in I} \beta_i w_i = h$, then $|\bar{h}|_{D \cup X} = |w|_{D \cup X} \geq 2$. We collect all such h to make a new set H_0 and denote $S_1 = S^{(0)} \cup H_0$. It is clear that each $h \in H_0$ is $D \cup X$ -homogeneous and we call $|\bar{h}|_{D \cup X}$ the $D \cup X$ -length of h . Now we reduce S_1 to a minimal set $S^{(1)}$. Noting that $S^{(0)}$ is a minimal set, if some inclusion composition is not trivial, then it must involve some element that is not in $S^{(0)}$. Furthermore, each $h \in H_0$ has $D \cup X$ -length at least 2, so every non-trivial inclusion composition that is added also has $D \cup X$ -length at least 2. So if we denote $S^{(1)} = S_{sub}^{(0)} \cup R^{(0)}$, where $S_{sub}^{(0)} = S^{(1)} \cap S^{(0)}$ and $R^{(0)} = S^{(1)} \setminus S^{(0)}$, then we get each $r \in R^{(0)}$, $|\bar{r}|_{D \cup X} \geq 2$. For any $f, g \in S_{sub}^{(0)}$, if

$$[D^{t_1}f, D^{t_2}g]_w \equiv 0 \mod(S_1, w),$$

then

$$[D^{t_1}f, D^{t_2}g]_w \equiv 0 \mod(S^{(1)}, w)$$

by Lemma 4.1. Continue this progress, and suppose

$$S_n = S^{(n-1)} \cup H_{n-1}, \quad S^{(n)} = S_{sub}^{(n-1)} \cup R^{(n-1)},$$

where $S^{(n)}$ is a minimal set and for any $h \in H_{n-1}$, $r \in R^{(n-1)}$, $|\bar{h}|_{D \cup X} \geq n+1$, $|\bar{r}|_{D \cup X} \geq n+1$. Then in order to get S_{n+1} , for any $f, g \in S^{(n)}$, $t_1, t_2 \in \mathbb{N}$, $t_1, t_2 \leq n+1$, $w = lcm(\overline{D^{t_1}f}, \overline{D^{t_2}g})$, we need to check composition $[D^{t_1}f, D^{t_2}g]_w$ whenever w is non-trivial. If all such compositions are trivial, we just set $S_{n+1} = S^{(n)}$; Otherwise, say $[D^{t_1}f, D^{t_2}g]_w = h$ is not trivial. If $f, g \in S_{sub}^{(n-1)} \subseteq S_n$, $0 \leq t_1, t_2 \leq n$, then by the construction, $[D^{t_1}f, D^{t_2}g]_w \equiv 0 \mod(S_n, w)$, and by Lemma 4.1, we get $[D^{t_1}f, D^{t_2}g]_w \equiv 0 \mod(S^{(n)}, w)$. Therefore, if we have a non-trivial composition, at least one of f and g is in $R^{(n-1)}$, or at least one of t_1 and t_2 equals $n+1$. Thus, if we denote $S_{n+1} = S^{(n)} \cup H_n$, then for any $h \in H_n$, $|\bar{h}|_{D \cup X} \geq n+2$. By the same reasoning as above, if we continue to reduce S_{n+1} to a minimal set $S^{(n+1)} = S_{sub}^{(n)} \cup R^{(n)}$, then we get for all $r \in R^{(n)}$, $|\bar{r}|_{D \cup X} \geq n+2$. As a result, we get the following lemma.

Lemma 4.2. *For any $f \in S^{(n)}$, if $|\bar{f}|_{D \cup X} \leq n$, then $f \in S^{(l)}$ for any $l \geq n$.*

Proof. Noting that after we get $S^{(n)}$, any composition that may be added afterwards has $D \cup X$ -length more than n , but f can not be reduced by any element which has $D \cup X$ -length more than n or by element in $S^{(n)} \setminus \{f\}$. ■

Define

$$\tilde{S} = \{f \in \bigcup_{n \geq 0} S^{(n)} \mid f \in S^{(|\bar{f}|_{D \cup X})}\}.$$

Then by Lemma 4.2, we have

$$\tilde{S} = \{f \in \bigcup_{n \geq 0} S^{(n)} \mid f \in S^{(l)}, \text{ for any } l \geq |\bar{f}|_{D \cup X}\}.$$

Lemma 4.3. $Id[S] = Id[\tilde{S}]$ and \tilde{S} is a Gröbner-Shirshov basis in $k\{X\}$.

Proof. Since $Id[S] = Id[S^{(0)}] = \dots = Id[S^{(n)}]$ for any $n \geq 0$, we have $Id[\tilde{S}] \subseteq Id[S]$. On the other hand, for any $f \in S$, if $|\bar{f}|_{D \cup X} = n$, then by Lemma 4.1, $f = \sum \beta_q u_q D^{t_q} s_q$, where each $s_q \in S^{(n)}$ and $|\bar{s}_q|_{D \cup X} \leq n$, i.e., $s_q \in \tilde{S}$. Therefore, $Id[S] = Id[\tilde{S}]$. For any $f, g \in \tilde{S}$, $t_1, t_2 \in \mathbb{N}$, $w = lcm(\overline{D^{t_1} f}, \overline{D^{t_2} g})$, let $l \triangleq |\bar{f}|_{D \cup X} + |\bar{g}|_{D \cup X} + t_1 + t_2$. If there exists composition $[D^{t_1} f, D^{t_2} g]_w$, then $[D^{t_1} f, D^{t_2} g]_w \equiv 0 \pmod{(S_{l+1}, w)}$ by construction. And by Lemma 4.1, we have $[D^{t_1} f, D^{t_2} g]_w \equiv 0 \pmod{(S^{(l+1)}, w)}$, i.e., $[D^{t_1} f, D^{t_2} g]_w = \sum \beta_i u_i D^{t_i} s_i$, where each $s_i \in S^{(l+1)}$ and $|\bar{s}_i|_{D \cup X} \leq |w|_{D \cup X} < l + 1$. Thus by the definition of \tilde{S} , we get $[D^{t_1} f, D^{t_2} g]_w \equiv 0 \pmod{(\tilde{S}, w)}$. \blacksquare

Proposition 4.1. If $|S| < \infty$ and S is $D \cup X$ -homogeneous, then $k\{X|S\}$ has a solvable word problem.

Proof. For any $f = \sum \beta_i w_i \in k\{X\}$, where $w_1 > w_2 > \dots$. We may assume that $|w_i|_{D \cup X} \leq n$. By Lemma 4.3 and Theorem 1, $f \in Id[\tilde{S}]$ implies that $w_1 = u \overline{D^t s}$ for some $s \in \tilde{S}$. Moreover, if $w_1 = u \overline{D^t s}$ for some $s \in \tilde{S}$, then $s \in S^{(n)}$. Note that $S^{(n)}$ is a finite $D \cup X$ -homogeneous set that can be constructed effectively from S . After reducing f by such s , we get a new polynomial

$$f' = f - \frac{\beta_1}{LC(u D^t s)} u D^t s = \sum \beta'_i w'_i,$$

with each $|w'_i|_{D \cup X} \leq n$. Continue to reduce f' by elements in $S^{(n)}$. If finally we reduce f' by $S^{(n)}$ to 0, then $f \in Id[S]$. Otherwise, $f \notin Id[S]$. In particular, if $w_1 \neq u \overline{D^t s}$ for any $s \in S^{(n)}, t \in \mathbb{N}$, then $w_1 \neq u \overline{D^t s}$ for any $s \in \tilde{S}, t \in \mathbb{N}$, and thus $f \notin Id[S]$. \blacksquare

Since for any $f \in GDN(X)$, if $f = \sum_{1 \leq i \leq n} \beta_i w_i$ is homogeneous in the sense that $|w_1| = \dots = |w_n|$, then f is $D \cup X$ -homogeneous because $|w|_{D \cup X} = 2|w| + wt(w)$ for any

$w \in [D^\omega X]$. Given $GDN(X|S)$, if $|S| < \infty$ and S is homogeneous, then taking S as a subset of $k\{X\}$, S is $D \cup X$ -homogeneous. Thus we can get a Gröbner-Shirshov basis \tilde{S} in $k\{X\}$. Then by Theorem 3 and Proposition 4.1, we immediately get the following proposition.

Proposition 4.2. *If $|S| < \infty$ and $S \subseteq GDN(X)$ is homogeneous, then $GDN(X|S)$ has a solvable word problem.*

5 A subalgebra of $GDN(a)$

We construct a non-free subalgebra A of the free Gelfand-Dorfman-Novikov algebra $GDN(a)$ over a field of characteristic 0, which implies that the variety of Gelfand-Dorfman-Novikov algebras is not Schreier.

By Proposition 1 in [13], we immediately get the following lemma.

Lemma 5.1. ([13]) *The following statements hold:*

- (i) *The rank of a free Gelfand-Dorfman-Novikov algebra is uniquely determined, where the rank means the number of free generators.*
- (ii) *In a free Gelfand-Dorfman-Novikov algebra of rank n , any set of n generators is a set of free generators.*
- (iii) *A free Gelfand-Dorfman-Novikov algebra of rank n can't be generated by less than n elements.*

In this subsection, we consider the free Gelfand-Dorfman-Novikov algebra $GDN(a)$ generated by one element a .

Theorem 4. *Let $A = \langle a \circ a, (a \circ a) \circ a, ((a \circ a) \circ a) \circ a \rangle$ be the subalgebra of the free Gelfand-Dorfman-Novikov algebra $GDN(a)$ generated by the set $\{a \circ a, (a \circ a) \circ a, ((a \circ a) \circ a) \circ a\}$. Then A is not free.*

Proof. Suppose that A is free. Then by Lemma 5.1 (iii), we get $\text{rank}(A) \leq 3$.

If $\text{rank}(A) = 3$, then by Lemma 5.1, $a \circ a, (a \circ a) \circ a, ((a \circ a) \circ a) \circ a$ are free generators. However,

$$(a \circ a) \circ (((a \circ a) \circ a) \circ a) = ((a \circ a) \circ a) \circ ((a \circ a) \circ a),$$

which means that $a \circ a, (a \circ a) \circ a, ((a \circ a) \circ a) \circ a$ are not free generators.

If $\text{rank}(A) = 1$ and

$$f = \beta_1(a \circ a) + \beta_2(a \circ a) \circ a + \sum \beta_i w_i$$

is a free generator of A , where each w_i has length at least 4, then

$$a \circ a = \gamma_1 f + \sum \gamma_j f \circ f \circ f \circ \cdots \circ f,$$

where f occurs at least twice in each term of the second summand on the right side and each of them is with some bracketing. We can rewrite this formula to the following form:

$$a[0]a[-1] = \gamma_1 f + \sum \lambda_{i_1, i_2, \dots, i_n} (D^{i_1} f)(D^{i_2} f) \cdots (D^{i_n} f),$$

where $i_1 \geq i_2 \geq \cdots \geq i_n \geq 0, n \geq 2$. Then each term in the second summand has leading term bigger than $a[0]a[-1]$. Since

$$\overline{(D^{i_1} f)(D^{i_2} f) \cdots (D^{i_n} f)} = \overline{(D^{i_1} f)} \overline{(D^{i_2} f)} \cdots \overline{(D^{i_n} f)},$$

by analysing the leading terms of the left side and the right side, we get each $\lambda_{i_1, i_2, \dots, i_n} = 0$, so $a[0]a[-1] = \gamma_1 f$, i.e., $f = \frac{1}{\gamma_1} a \circ a$. However, $(a \circ a) \circ a \notin \langle a \circ a \rangle = A$. This is a contradiction.

If $\text{rank}(A) = 2$, suppose

$$f_1 = \beta_1(a \circ a) + \beta_2((a \circ a) \circ a) + \sum \beta_e w_e,$$

$$f_2 = \gamma_1(a \circ a) + \gamma_2((a \circ a) \circ a) + \sum \gamma_{e'} w_{e'},$$

are free generators, where each $w_e, w_{e'}$ has length at least 4. Say

$$a \circ a = \lambda_1 f_1 + \lambda_2 f_2 + \sum \lambda_{j_1, j_2, \dots, j_n} f_{j_1} \circ f_{j_2} \circ \cdots \circ f_{j_n},$$

$$(a \circ a) \circ a = \mu_1 f_1 + \mu_2 f_2 + \sum \mu_{q_1, q_2, \dots, q_m} f_{q_1} \circ f_{q_2} \circ \cdots \circ f_{q_m},$$

$$((a \circ a) \circ a) \circ a = \nu_1 f_1 + \nu_2 f_2 + \sum \nu_{l_1, l_2, \dots, l_r} f_{l_1} \circ f_{l_2} \circ \cdots \circ f_{l_r},$$

where $j_1, \dots, j_n, q_1, \dots, q_m, l_1, \dots, l_r \in \{1, 2\}$, and each term in the third summand on the right side of each equation is with some bracketing. Rewriting the right sides into linear combination of basis of the free Gelfand-Dorfman-Novikov algebra $GDN(a)$ and comparing terms of length 2 and 3 on the left sides and the right sides, we get

$$\begin{pmatrix} \beta_1 & \gamma_1 \\ \beta_2 & \gamma_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} \beta_1 & \gamma_1 \\ \beta_2 & \gamma_2 \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So $\nu_1 = \nu_2 = 0$ and

$$((a \circ a) \circ a) \circ a = \sum \nu_{l_1, l_2, \dots, l_r} f_{l_1} \circ f_{l_2} \circ \dots \circ f_{l_r}.$$

However, among the terms of the right side, only $(a \circ a) \circ (a \circ a)$ has length 4, but $((a \circ a) \circ a) \circ a \neq \beta(a \circ a) \circ (a \circ a)$, for any $\beta \in k$.

Therefore, A is not free. ■

Acknowledgment

We would like to thank the referee for valuable suggestions.

References

- [1] C. Bai and D. Meng, The classification of Novikov algebras in low dimensions, *J. Phys. A* **34** (2001) 1581-1594.
- [2] C. Bai and D. Meng, Addendum: The classification of Novikov algebras in low dimensions: invariant bilinear forms, *J. Phys. A* **34** (2001) 8193-8197.
- [3] C. Bai and D. Meng, Transitive Novikov algebras on four-dimensional nilpotent Lie algebras, *Int. J. Theoret. Phys.* **40** (2001) 1761-1768.

- [4] A.A. Balinskii and S.P. Novikov, Poisson brackets of hydrodynamic type, Frobenius algebras and Lie algebras, *Dokl. Akad. Nauk SSSR* **283** (1985) 1036-1039.
- [5] L.A. Bokut, Insolvability of the word problem for Lie algebras, and subalgebras of finitely presented Lie algebras, *Izvestija AN USSR (mathem.)* **36** (1972) 1173-1219.
- [6] L.A. Bokut and Yuqun Chen, Gröbner-Shirshov bases and their calculation, *Bull. Math. Sci.* **4** (2014) 325-395.
- [7] L.A. Bokut, S.-J. Kang, K.-H. Lee and P. Malcolmson, Gröbner-Shirshov bases for Lie superalgebras and their universal enveloping algebras, *J. Alg.* **217** (1999) 461-495.
- [8] L.A. Bokut and P. Malcolmson, Gröbner-Shirshov bases for Lie and associative algebras, Collection of Abstracts, Shum, Kar-Ping (ed.) et al., Algebras and combinatorics, Papers from the international congress, ICAC'97, Hong Kong, August 1997. Singapore: Springer, (1999) 139-142.
- [9] L.A. Bokut and P. Malcolmson, Gröbner-Shirshov bases for relations of a Lie algebra and its enveloping algebra, Shum, Kar-Ping (ed.) et al., Algebras and combinatorics, Papers from the international congress, ICAC'97, Hong Kong, August 1997. Singapore: Springer, (1999) 47-54.
- [10] B. Buchberger, An algorithmical criteria for the solvability of algebraic systems of equations, *Aequationes Math.* **4** (1970) 374-383.
- [11] D. Burde and K. Dekimpe, Novikov structures on solvable Lie algebras, *J. Geom. Phys.* **56** (2006) 1837-1855.
- [12] D. Burde and W. Graaf, Classification of Novikov algebras, *Appl. Algebra Engrg. Comm. Comput.* **24** (2013) 1-15.
- [13] M.S. Burgin and V.A. Artamonov, Some property of subalgebras in varieties of linear Ω -algebras, *Math. USSR Sbornik* **16** (1972) 69-85.
- [14] L. Chen, Y. Niu and D. Meng, Two kinds of Novikov algebras and their realizations, *J. Pure Appl. Alg.* **212** (2008) 902-909.

- [15] Askar Dzhumadil'daev, Codimension growth and non-koszulity of Novikov operad, *Commun. Alg.* **39** (2011) 2943-2952.
- [16] Askar Dzhumadil'daev and Clas Löfwall, Trees, free right-symmetric algebras, free Novikov algebras and identities, *Homology, Homotopy and Applications* **4** (2002) 165-190.
- [17] G. Carrà Ferro, A survey on differential Gröbner bases, *Radon Series Comp. Appl. Math.* **2** (2007) 77-108.
- [18] V.T. Filippov, A class of simple nonassociative algebras, *Mat. Zametki* **45** (1989) 101-105.
- [19] V.T. Filippov, On right-symmetric and Novikov nil-algebras of bounded index, (Russian) *Mat. Zametki* **70** (2001) 289-295.
- [20] I.M. Gelfand and I.Ya. Dorfman, Hamiltonian operators and algebraic structures related to them, *Funkts. Anal. Prilozhen* **13** (1979) 13-30.
- [21] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero, I, II, *Ann. of Math.* **79** (1964) 109-203, 205-326.
- [22] E.R. Kolchin, *Differential Algebra and Algebraic Groups* (New York: Academic, 1973).
- [23] M.V. Kondratieva and A.I. Zobnin, Membership problem for differential ideals generated by a composition of polynomials, ISSN 0361-7688, Programming and Computer Software **32** (2006) 123-127.
- [24] A.G. Kurosh, Nonassociative free algebras and free products of algebras, *Mat. Sb.*, **20** (1947) 119-126.
- [25] Leonid Makar-Limanov and Ualbai Umirbaev, The Freiheitssatz for Novikov algebras, *TWMS J. Pure Appl. Math.* **2** (2011) 228-235.
- [26] S.P. Novikov, Geometry of conservative systems of hydrodynamic type, The averaging method for field-theoretic systems, (Russian) International conference on current

- problems in algebra and analysis (Moscow-Leningrad, 1984) *Uspekhi Mat. Nauk* **40** (1985).
- [27] F. Ollivier, Standard bases of differential ideals, Lect. Notes. in Computer Science **508** (1990) 304-321.
 - [28] J.M. Osborn, Novikov algebras, *Nova J. Alg. Geom.* **1** (1992) 1-13.
 - [29] J.M. Osborn, Simple Novikov algebras with an idempotent, *Commun. Alg.* **20** (1992) 2729-2753.
 - [30] J.M. Osborn, Infinite dimensional Novikov algebras of characteristic 0, *J. Alg.* **167** (1994) 146-167.
 - [31] J.M. Osborn, Modules over Novikov algebras of characteristic 0, *Commun. Alg.* **23** (1995) 3627-3640.
 - [32] J.M. Osborn and E.I. Zelmanov, Nonassociative algebras related to Hamiltonian operators in the formal calculus of variations, *J. Pure. Appl. Algebra* **101** (1995) 335-352.
 - [33] O.J. Schreier, Die Untergruppen der freien Gruppen, *Abh. Math., Sem. Univ. Hamburg* **5** (1927) 161-183.
 - [34] Anne V. Shepler and Sarah Witherspoon, A Poincaré-Birkhoff-Witt theorem for quadratic algebras with group actions, *Trans. Amer. Math. Soc.* **366** (2014) 6483-6506.
 - [35] A.I. Shirshov, Subalgebras of free Lie algebras, *Mat. Sb.*, **33** (1953) 441-452.
 - [36] A.I. Shirshov, Some algorithmic problems for ε -algebras, *Sibirsk. Mat. Zh.* **3** (1962) 132-137.
 - [37] A.I. Shirshov, Some algorithmic problems for Lie algebras, *Sibirsk. Mat. Zh.* **3** (1962) 292-296 (in Russian). English translation: *SIGSAM Bull.* **33** (1999) 3-6.
 - [38] Selected works of A.I. Shirshov, Eds L.A. Bokut, V. Latyshev, I. Shestakov, E.I. Zelmanov, Trs M. Bremner, M. Kochetov, Birkhäuser, Basel, Boston, Berlin, 2009.

- [39] U.U. Umirbaev, Schreier varieties of algebras, *Algebra and Logic*, **33** (1994) 180-193.
- [40] E. Witt, Die Unterringe der freien Lieschen Ring, *Math. Z.*, **64** (1956) 195-216.
- [41] X. Xu, Hamiltonian operators and associative algebras with a derivation, *Lett. Math. Phys.* **33** (1995) 1-6.
- [42] X. Xu, On simple Novikov algebras and their irreducible modules, *J. Alg.* **185** (1996) 905-934.
- [43] X. Xu, Novikov-Poisson algebras, *J. Alg.* **190** (1997) 253-279.
- [44] X. Xu, Variational calculus of supervariables and related algebraic structures, *J. Alg.* **223** (2000) 396-437.
- [45] E.I. Zelmanov, On a class of local translation invariant Lie algebras, *Dokl. Akad. Nauk SSSR* **292** (1987) 1294-1297.
- [46] E.I. Zelmanov, Engelian Lie algebras, (Russian) *Sibirsk. Mat. Zhur.* **29** (1988) 112-117.
- [47] E.I. Zelmanov, Engelian Lie Algebras, *Siberian Math. J.* **29** (1989) 777-781.
- [48] A.I. Zhukov, Complete systems of defining relations in noassociative algebras, *Mat. Sbornik* **69** (1950) 267-280.